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A geometric approach to constrained mechanical systems, symmetries and inverse problems

Paola Morando†§ and Stefano Vignolo‡‖

† Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

‡ Dipartimento di Matematica, Universitá degli Studi di Genova Via Dodecaneso 35, 16146 Genova, Italy

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Abstract. We introduce a geometrical framework for the description of constrained mechanical systems, and we analyse different kinds of symmetries and their relationships. We propose a new definition for non-holonomic Lagrangian mechanical systems, and we give a geometrical characterization for the Helmholtz conditions related to the inverse problem.

0. Introduction

In recent papers [4–6, 9–11] various frameworks for the description of non-holonomic mechanical systems were proposed. In this paper some well known results valid for second-order differential equations (SODE) representing the dynamical evolution of a non-constrained mechanical system will be extended to the case of constrained mechanical systems, making use of the formalism introduced, e.g. in [9–11].

We consider the jet bundle $J^1(E)$, where $\tau_0 : E \to \mathcal{R}$ is a fibred bundle on the real line $\mathcal{R}$, and a fibred submanifold Σ of $J^1(E)$ describing the kinetic constraints. It is well known that a system of SODE may be described by a vector field $\tilde{\Gamma}$ on $J^1(E)$, henceforth called the non-constrained SODE. In a similar way, a mechanical system with kinetic constraints may be described by a vector field Γ on $J^1(E)$, tangent to the manifold Σ. We propose an analysis of the symmetries of a constrained SODE Γ based on a generalization of correspondent results holding for the non-constrained case. A geometrical interpretation of new conditions arising will also be presented.

In section 3 we propose a definition of a Lagrangian mechanical system with constraints. In the holonomic case, given a mechanical system with Lagrangian function $\tilde{\boldsymbol{L}} \in \mathcal{C}^\infty(J^1(E))$, it is well known that the equations of motion can be written as $i_{\tilde{\Gamma}}\mathrm{d}\Theta = 0$, where Θ is the Poincaré Cartan 1-form associated with $\tilde{\boldsymbol{L}}$. In the presence of kinetic constraints, the equations of motion for the mechanical system cannot be completely determined by a Lagrangian function $\boldsymbol{L}$, defined on the constraint's submanifold Σ.

Generalizing the situation when the constrained SODE Γ is obtained by projecting on Σ a Lagrangian SODE $\tilde{\Gamma}$ on $J^1(E)$, we define a 'non-holonomic Lagrangian' for a constrained SODE as a couple $(\boldsymbol{L}, \mu)$, where $\boldsymbol{L}$ is a function defined on Σ, and μ is a 1-form playing the role of the canonical momenta.

§ E-mail address: morando@polito.it

‖ E-mail address: vignolo@dima.unige.it

With this definition, the inverse problem for a constrained SODE may be formulated as follows. A SODE Γ on Σ will be said to be Lagrangian, with associated non-holonomic Lagrangian $(\boldsymbol{L}, \mu)$ if there exists a 1-form $\Theta = \frac{\partial \boldsymbol{L}}{\partial \dot{q}^\alpha}\theta^\alpha + \mu + \boldsymbol{L}\mathrm{d}t$ (analogous to the Poincaré Cartan 1-form) such that the equations of motion can be written as $i_\Gamma(\mathrm{d}\Theta) \in \mathrm{Span}\{\eta^a\}$, where the 1-forms η^a will be defined in section 1.

Generalizing some results of [1], we give a geometrical version of the Helmholtz conditions for a constrained SODE. In particular, we examine a particular class of constrained SODEs (satisfying $i_\Gamma(\mathrm{d}\Theta) = 0$), and extend some results valid in the non-constrained case.

Finally, we discuss how the geometrical framework introduced may be applied to the study of mixed first- and second-order systems of differential equations, along the lines proposed in [12].

We conclude the paper with an illustrative example of a Lagrangian mechanical system with constraints in the sense introduced in section 3.

1. Preliminaries

A suitable geometrical framework for the study of the evolution of a mechanical system subjected to (non-integrable) kinetic constraints is based on the introduction of a fibre bundle $\pi : E \to M$, where E and M are both fibre bundle over the real line $\mathcal{R}$.

Introducing fibred coordinates on E and M of the form $(t, q^\alpha, q^a) = (t, q^A)$ and (t, q^α) where $\alpha = 1 \ldots r, a = 1 \ldots n - r$ and $A = 1 \ldots n$, let us consider the fibred product $\tilde{\Sigma} = E \times_M J^1(M)$ with projections and $\pi_1 : \tilde{\Sigma} \to E$ and $\pi_2 : \tilde{\Sigma} \to J^1(M)$.

If $J^1(E)$ is the first jet extension of the fibre bundle E with respect to the fibration $\tau_0 : E \to \mathcal{R}$, let us denote by $i : \tilde{\Sigma} \to J^1(E)$ the injection described in local coordinates by

$$(t, q^A, \dot{q}^\alpha) \to (t, q^A, \dot{q}^\alpha, g^a(t, q^A, \dot{q}^\alpha)).$$

By means of i the manifold $\tilde{\Sigma}$ may be identified with a fibred submanifold Σ of $J^1(E)$, described by equations $\dot{q}^a = g^a(t, q^A, \dot{q}^\alpha)$, which will be henceforth referred as the constraint's manifold. With a slight abuse of notation we will also denote by i the inclusion map $i : \Sigma \to J^1(E)$.

According to this identification, a section $\gamma : \mathcal{R} \to E$ will be said to be *kinematically admissible* iff $j^1(\gamma) \in \Sigma$, where $j^1(\gamma)$ is the first jet extension of γ.

Keeping the same notations as in [9, 10], to every SODE on Σ we associate a corresponding vector field on Σ of the form

$$\Gamma = \frac{\partial}{\partial t} + \dot{q}^\alpha \frac{\partial}{\partial q^\alpha} + g^a(t, q^A, \dot{q}^\beta)\frac{\partial}{\partial q^a} + f^\alpha(t, q^A, \dot{q}^\alpha)\frac{\partial}{\partial \dot{q}^\alpha}. \tag{1}$$

It is easy to see that each integral curve of Γ is the jet-extension of a kinematically admissible section of E.

The fibre bundle $J^1(M)$ carries the action of the well known canonical endomorphism S [2], expressed in coordinates as the $(1, 1)$-type tensor field

$$S = \theta^\alpha \otimes \frac{\partial}{\partial \dot{q}^\alpha}$$

where $\theta^\alpha = \mathrm{d}q^\alpha - \dot{q}^\alpha \mathrm{d}t$ are the canonical contact 1-forms on $J^1(M)$. It is easy to check that S is a well-defined tensor field also on Σ.

It is well known [9, 10] that the the eigenspaces of the tensor field $\dot{S} := \mathcal{L}_\Gamma S$ induce a decomposition of the tangent bundle $T\Sigma$ and of the cotangent bundle $T^*\Sigma$ into direct sum of subbundles.

A local canonical basis on $T\Sigma$, adapted to the stated decomposition, is provided by the vectors

$$\Gamma \qquad H_\alpha = \frac{\partial}{\partial q^\alpha} + B^a_\alpha \frac{\partial}{\partial q^a} - \Gamma^\beta_\alpha \frac{\partial}{\partial \dot{q}^\beta}, \frac{\partial}{\partial q^a}, \frac{\partial}{\partial \dot{q}^\alpha} \tag{2}$$

with

$$B^a_\alpha = \frac{\partial g^a}{\partial \dot{q}^\alpha} \qquad \Gamma^\alpha_\beta = -\frac{1}{2}\frac{\partial f^\alpha}{\partial \dot{q}^\beta}.$$

The correspondent dual basis in $T^*\Sigma$ consists of the 1-forms

$$\mathrm{d}t, \theta^\alpha, \eta^a = \mathrm{d}q^a - g^a \mathrm{d}t - B^a_\alpha \theta^\alpha, \phi^\alpha = \mathrm{d}\dot{q}^\alpha - f^\alpha \mathrm{d}t + \Gamma^\alpha_\beta \theta^\beta. \tag{3}$$

We remark that the 1-forms η^a span locally a codistribution intrinsically associated to the constraint manifold. This is the Chetaev bundle recently introduced by several authors (see, e.g. [6, 13]). It is immediate to verify that the 1-forms θ^α and η^a generate locally the contact bundle over Σ $\mathcal{C}(\Sigma) = i^*(\mathcal{C}(J^1E))$, where $\mathcal{C}(J^1E)$ is the contact bundle over $J^1(E)$ spanned locally by $\theta^A = \mathrm{d}q^A - \dot{q}^A \mathrm{d}t$.

In terms of the bases (2) and (3) we have the representation

$$\dot{S} = -H_\alpha \otimes \theta^\alpha + \frac{\partial}{\partial \dot{q}^\alpha} \otimes \phi^\alpha.$$

2. Symmetries

The canonical endomorphism S on Σ allows us to introduce an almost product structure on Σ, i.e. a $(1, 1)$-type tensor field A such that $A^2 = I$, which is given in local coordinates by

$$A = \dot{S} + \Gamma \otimes \mathrm{d}t + \frac{\partial}{\partial q^a} \otimes \eta^a.$$

It is easy to see that A is an automorphism of $\mathcal{D}^1(\Sigma)$ and of $\mathcal{D}_1(\Sigma)$ (the moduli of vector fields and 1-forms on Σ respectively), which preserves (up to the sign) the bases given by (2) and (3).

Moreover, we will consider the action of A on the whole tensor algebra on Σ, as the action of tensor product $A \otimes A \cdots \otimes A$. In particular we have

(1) $A(f) = f, \forall f \in \mathcal{C}^\infty(\Sigma)$

(2) $A(U \otimes W) = AU \otimes AW \forall U, W$ tensor on Σ.

Definition 2.1. We denote by $\mathcal{A}_\Gamma$ the differential operator acting on tensor fields over Σ as

$$\mathcal{A}_\Gamma = A\mathcal{L}_\Gamma A \tag{4}$$

where $\mathcal{L}_\Gamma$ is the Lie derivative along Γ.

Remark 2.1.

(i) $\mathcal{A}_\Gamma$ is a derivation of degree zero commuting with contractions,

(ii) $\mathcal{A}_\Gamma(f) = \Gamma(f), \forall f \in \mathcal{C}^\infty(\Sigma)$,

(iii) $\mathcal{L}_\Gamma = A\mathcal{A}_\Gamma A$.

Definition 2.2. Let Γ be a SODE on Σ, and let $X \in \mathcal{D}^1(\Sigma)$ and $\sigma \in \mathcal{D}_1(\Sigma)$ be a vector field and a 1-form on Σ respectively. Then

(i) X is a dynamical symmetry for Γ iff $\mathcal{L}_\Gamma X = h\Gamma$, where $h \in \mathcal{C}^\infty(\Sigma)$,
(ii) X is a dual-adjoint symmetry for Γ iff $\mathcal{A}_\Gamma X = h\Gamma$, where $h \in \mathcal{C}^\infty(\Sigma)$,
(iii) σ is an adjoint symmetry of Γ iff $\mathcal{A}_\Gamma \sigma = h\mathrm{d}t$, where $h \in \mathcal{C}^\infty(\Sigma)$,
(iv) σ is a dual symmetry of Γ iff $\mathcal{L}_\Gamma \sigma = h\mathrm{d}t$, where $h \in \mathcal{C}^\infty(\Sigma)$.

In terms of the bases (2) and (3) a vector field $X = X^0\Gamma + X^\alpha H_\alpha + X^a \frac{\partial}{\partial q^a} + \bar{X}^\alpha \frac{\partial}{\partial \dot{q}^\alpha}$ is a dynamical symmetry of Γ iff it satisfies the conditions

$$\Gamma(X^\alpha) + X^\beta \Gamma^\alpha_\beta - \bar{X}^\alpha = 0 \tag{5a}$$

$$\Gamma(X^a) - X^b \frac{\partial g^a}{\partial q^b} + X^\alpha Q^a_\alpha = 0 \tag{5b}$$

$$\Gamma(\bar{X}^\alpha) + \bar{X}^\beta \Gamma^\alpha_\beta + X^\beta \phi^\alpha_\beta - X^a \frac{\partial f^\alpha}{\partial q^a} = 0 \tag{5c}$$

where $Q^a_\alpha = \Gamma(B^a_\alpha) - \frac{\partial g^a}{\partial q^\alpha} - B^b_\alpha \frac{\partial g^a}{\partial q^b}$, and $\phi^\beta_\alpha = -H_\alpha(f^\beta) + \Gamma^\beta_\gamma \Gamma^\gamma_\alpha - \Gamma(\Gamma^\beta_\alpha)$.

In a similar way, a 1-form $\sigma = \sigma_0 \mathrm{d}t + \sigma_\alpha \theta^\alpha + \sigma_a \eta^a + \bar{\sigma}_\alpha \phi^\alpha$ is an adjoint symmetry if it satisfies the following conditions:

$$\Gamma(\bar{\sigma}_\alpha) - \bar{\sigma}_\beta \Gamma^\beta_\alpha - \sigma_\alpha = 0 \tag{6a}$$

$$\Gamma(\sigma_a) + \sigma_b \frac{\partial g^b}{\partial q^a} + \bar{\sigma}_\alpha \frac{\partial f^\alpha}{\partial q^a} = 0 \tag{6b}$$

$$\Gamma(\sigma_\alpha) - \sigma_\beta \Gamma^\beta_\alpha + \sigma_a Q^a_\alpha + \bar{\sigma}_\beta \phi^\beta_\alpha = 0. \tag{6c}$$

An analogous expression can be obtained for dual-adjoint and dual symmetries.

Proceeding as in [7], we can define four subsets of $\mathcal{D}^1(\Sigma)$ and $\mathcal{D}_1(\Sigma)$ respectively. In terms of the operators $\mathcal{L}_\Gamma$ and $\mathcal{A}_\Gamma$ these are defined as follows:

$$\mathcal{X}_\Gamma = \left\{ X \in \mathcal{D}^1(\Sigma) | X = \mathcal{A}_\Gamma \left(X^\alpha \frac{\partial}{\partial \dot{q}^\alpha} \right) + Y, Y \in \mathrm{Span} \left\{ \Gamma, \frac{\partial}{\partial q^a} \right\} \right\}$$

$$\mathcal{M}_\Gamma = \left\{ X \in \mathcal{D}^1(\Sigma) | X = \mathcal{L}_\Gamma \left(X^\alpha \frac{\partial}{\partial \dot{q}^\alpha} \right) + Y, Y \in \mathrm{Span} \left\{ \Gamma, \frac{\partial}{\partial q^a} \right\} \right\}$$

$$\mathcal{M}^\star_\Gamma = \{ \sigma \in \mathcal{D}_1(\Sigma) | \sigma = \mathcal{L}_\Gamma(\sigma_\alpha \theta^\alpha) + \nu, \nu \in \mathrm{Span}\{\mathrm{d}t, \eta^a\} \}$$

$$\mathcal{X}^\star_\Gamma = \{ \sigma \in \mathcal{D}_1(\Sigma) | \sigma = \mathcal{A}_\Gamma(\sigma_\alpha \theta^\alpha) + \nu, \nu \in \mathrm{Span}\{\mathrm{d}t, \eta^a\} \}.$$

In local coordinates we have the equivalent characterizations

$$X \in \mathcal{X}_\Gamma \iff X = X^0 \Gamma + X^\alpha H_\alpha + X^a \frac{\partial}{\partial q^a} + (\Gamma(X^\alpha) + X^\beta \Gamma^\alpha_\beta) \frac{\partial}{\partial \dot{q}^\alpha}$$

$$\sigma \in \mathcal{M}^\star_\Gamma \iff \sigma = \sigma_0 \mathrm{d}t + (\Gamma(\sigma_\alpha) - \sigma_\beta \Gamma^\beta_\alpha) \theta^\alpha + \sigma_a \eta^a + \sigma_\alpha \phi^\alpha.$$

Theorem 2.1. Let Γ be a SODE on Σ, and let X and σ be a vector field and a 1-form on Σ respectively. Then

$$X \in \mathcal{X}_\Gamma \iff \mathcal{L}_\Gamma X \in \mathrm{Span} \left\{ \Gamma, \frac{\partial}{\partial q^a}, \frac{\partial}{\partial \dot{q}^\alpha} \right\}$$

$$X \in \mathcal{M}_\Gamma \iff \mathcal{A}_\Gamma X \in \mathrm{Span} \left\{ \Gamma, \frac{\partial}{\partial q^a}, \frac{\partial}{\partial \dot{q}^\alpha} \right\}$$

$$\sigma \in \mathcal{M}^\star_\Gamma \iff \mathcal{A}_\Gamma \sigma \in \mathrm{Span}\{\mathrm{d}t, \theta^\alpha, \eta^a\}$$

$$\sigma \in \mathcal{X}^\star_\Gamma \iff \mathcal{L}_\Gamma \sigma \in \mathrm{Span}\{\mathrm{d}t, \theta^\alpha, \eta^a\}.$$

Proof. The computation is entirely straightforward and is left to the reader. □

Corollary 2.1. If a vector field X is a dynamical symmetry (dual-adjoint symmetry), then $X \in \mathcal{X}_\Gamma$ $(X \in \mathcal{M}_\Gamma)$, and if a 1-form σ is an adjoint symmetry (dual symmetry), then $\sigma \in \mathcal{M}^\star_\Gamma (\sigma \in \mathcal{X}^\star_\Gamma)$.

Theorem 2.2. The tensor field A gives a bijection between $\mathcal{X}_\Gamma$ and $\mathcal{M}_\Gamma$, and between $\mathcal{X}^\star_\Gamma$ and $\mathcal{M}^\star_\Gamma$.

Proof. By definition $A^2 = I$ and A acts as the identity on Span$\{\Gamma, \frac{\partial}{\partial q^a}, \frac{\partial}{\partial \dot{q}^\alpha}\}$. Hence, recalling theorem 2.1, we have that $X \in \mathcal{X}_\Gamma \Longleftrightarrow \mathcal{L}_\Gamma X \in$ Span$\{\Gamma, \frac{\partial}{\partial q^a}, \frac{\partial}{\partial \dot{q}^\alpha}\} \Longleftrightarrow \mathcal{A}_\Gamma A(X) \in$ Span$\{\Gamma, \frac{\partial}{\partial q^a}, \frac{\partial}{\partial \dot{q}^\alpha}\} \Longleftrightarrow A(X) \in \mathcal{M}_\Gamma$. Similarly for $\mathcal{X}^\star_\Gamma$ and $\mathcal{M}^\star_\Gamma$. □

Theorem 2.3. The tensor field A gives a bijection between dynamical and dual-adjoint symmetries, and between dual and adjoint symmetries.

Proof. By definition A acts as the identity on the Span$\{\Gamma\}$, and $A\mathcal{L}_\Gamma = \mathcal{A}_\Gamma A$. Then, if X is a dynamical symmetry $A\mathcal{L}_\Gamma(X) = \mathcal{A}_\Gamma(AX) = h\Gamma$ with $h \in \mathcal{C}^\infty(\Sigma)$, and AX is a dual-adjoint symmetry. An entirely similar argument prove that if X is a dual-adjoint symmetry, then AX is a dynamical symmetry. The proof for adjoint and dual symmetry follows the same line. □

We recall that, if we consider a non-constrained SODE, the study of dynamical and adjoint symmetries gives rise to a pair of equations: the first one is an algebraic equation, playing the same role of (5*a*) and (6*a*), and the second one is a SODE, analogous to (5*c*) and (6*c*), the latter usually called the Jacobi equation associated to the SODE ([7, 8]). These conditions, both in the non-constrained and constrained cases, are related to the sets $\mathcal{X}_\Gamma$, $\mathcal{M}_\Gamma$, $\mathcal{M}^\star_\Gamma$, and $\mathcal{X}^\star_\Gamma$ through the following.

Theorem 2.4. Let Γ be a SODE on Σ, X a vector field and σ a 1-form on Σ. Then
$X \in \mathcal{X}_\Gamma$ and $\mathcal{L}_\Gamma(X) \in \mathcal{M}_\Gamma \Longleftrightarrow X$ satisfies conditions (5*a*) and (5*c*).
$\sigma \in \mathcal{M}^\star_\Gamma$ and $\mathcal{L}_\Gamma(\sigma) \in \mathcal{X}^\star_\Gamma \Longleftrightarrow \sigma$ satisfies conditions (6*a*) and (6*c*).

Proof. For non-constrained SODE the results may be found in [7, 8]. The extension to constrained case is similar, and follows from easy computations. □

We now turn to equations (5*b*) and (6*b*). These have no holonomic counterpart, but are typical of the non-holonomic case. To understand their meaning we have to introduce a wider geometrical framework.

Let $T_\Sigma(J^1(E))$ denote the restriction of $T(J^1(E))$ to the submanifold Σ, and let $\mathcal{D}^1_\Sigma(J^1(E))$ be the module of vector fields $X : \Sigma \to T_\Sigma(J^1(E))$. We decompose $T_\Sigma(J^1(E))$ into the direct sum

$$T_\Sigma(J^1(E)) = T\Sigma \oplus \mathcal{V}_\Sigma$$

where $\mathcal{V}_\Sigma$ is the vertical bundle spanned by $\{\frac{\partial}{\partial \dot{q}^a}\}$.

In local coordinates, for any vector field $X = X^0 \frac{\partial}{\partial t} + X^A \frac{\partial}{\partial q^A} + \bar{X}^A \frac{\partial}{\partial \dot{q}^A}$, $A = 1 \ldots n$, we have

$$X = i_* P(X) + Q(X)$$

where i denotes the inclusion map $i : \Sigma \to J^1(E)$, and

$$P : T_\Sigma(J^1(E)) \to T\Sigma \qquad Q : T_\Sigma(J^1(E)) \to \mathcal{V}_\Sigma$$

denote the linear operators having expression

$$P(X) = X^0 \frac{\partial}{\partial t} + X^A \frac{\partial}{\partial q^A} + \bar{X}^\alpha \frac{\partial}{\partial \dot{q}^\alpha}$$

$$Q(X) = \left(\bar{X}^a - X^0 \frac{\partial g^a}{\partial t} - X^\alpha \frac{\partial g^a}{\partial q^\alpha} - X^b \frac{\partial g^a}{\partial q^b} - \bar{X}^\alpha \frac{\partial g^a}{\partial \dot{q}^\alpha} \right) \frac{\partial}{\partial \dot{q}^a}.$$

We now introduce the set

$$\tilde{\mathcal{X}}_\Gamma(J^1(E)) = \left\{ X \in \mathcal{D}^1_\Sigma(J^1(E)) | X = X^A \frac{\partial}{\partial q^A} + \Gamma(X^A) \frac{\partial}{\partial \dot{q}^A} \right\}.$$

If we handle with a non-constrained SODE $\tilde{\Gamma}$, the set $\mathcal{X}_{\tilde{\Gamma}}$, analogous of $\tilde{\mathcal{X}}_\Gamma$, was introduced in [7], and a necessary condition for a vector field $X \in \mathcal{D}^1(J^1(E))$ to be a dynamical symmetry of $\tilde{\Gamma}$ is $X \in \mathcal{X}_{\tilde{\Gamma}}$.

For a constrained SODE we can give a geometrical interpretation of conditions (5*a*) and (5*b*) through the following theorem.

Theorem 2.5. Let Γ be a SODE on Σ and let $X \in \mathcal{D}^1_\Sigma(J^1(E))$ be a vector field such that $i_X \mathrm{d}t = 0$. Then $P(X)$ satisfies conditions (5*a*) and (5*b*) iff $X \in \tilde{\mathcal{X}}_\Gamma(J^1(E))$ and X is tangent to Σ.

Proof. The condition $X \in \tilde{\mathcal{X}}_\Gamma(J^1(E))$ is trivially equivalent to the fact that $P(X)$ satisfies (5*a*).

Moreover, the request that X be tangent to Σ, i.e.

$$\left(X^A \frac{\partial}{\partial q^A} + \Gamma(X^A) \frac{\partial}{\partial \dot{q}^A} \right) (\dot{q}^a - g^a(t, q^A, \dot{q}^\alpha)) = 0$$

gives the equation

$$\Gamma(X^a) - X^b \frac{\partial g^a}{\partial q^b} - X^\alpha \frac{\partial g^a}{\partial q^\alpha} - \Gamma(X^\alpha) \frac{\partial g^a}{\partial \dot{q}^\alpha} = 0$$

equivalent to (5*b*) for the vector field $P(X)$. □

3. The Lagrangian case

The aim of this section is to introduce a Lagrangian formalism in the study of constrained mechanical systems.

As a starting point, we consider a free (non-constrained) mechanical system, with a Lagrangian function $\tilde{\boldsymbol{L}} \in \mathcal{C}^\infty(J^1(E))$ satisfying the usual regularity condition

$$\det \left(\frac{\partial^2 \tilde{\boldsymbol{L}}}{\partial \dot{q}^A \partial \dot{q}^B} \right) \neq 0$$

and we introduce the corresponding Poincaré Cartan 1-form

$$\Theta_{\tilde{L}} = \frac{\partial \tilde{\boldsymbol{L}}}{\partial \dot{q}^A} \theta^A + \tilde{\boldsymbol{L}}\, \mathrm{d}t$$

where, as usual, $\theta^A = \mathrm{d}q^A - \dot{q}^A\, \mathrm{d}t$.

Then, we impose the kinetic constraints described by the manifold Σ.

Considering the restriction $i^*(\Theta_{\tilde{L}})$ of the Poincaré Cartan 1-form to Σ, it is possible to write the equations of motion for the constrained system in the form (see, e.g. [9, 11])

$$\Gamma\left(\frac{\partial \boldsymbol{L}}{\partial \dot{q}^\alpha}\right) - \frac{\partial \boldsymbol{L}}{\partial q^\alpha} - B^a_\alpha \frac{\partial \boldsymbol{L}}{\partial q^a} = i^*\left(\frac{\partial \tilde{\boldsymbol{L}}}{\partial \dot{q}^a}\right) Q^a_\alpha \tag{7}$$

where $\boldsymbol{L} = i^*(\tilde{\boldsymbol{L}})$ denotes the pull back of $\tilde{\boldsymbol{L}}$ on Σ, and

$$Q^a_\alpha = \Gamma(B^a_\alpha) - \frac{\partial g^a}{\partial q^\alpha} - B^b_\alpha \frac{\partial g^a}{\partial q^b}.$$

Under the regularity assumption

$$\det\left(\frac{\partial^2 \boldsymbol{L}}{\partial \dot{q}^\beta \partial \dot{q}^\alpha} - i^*\left(\frac{\partial \tilde{\boldsymbol{L}}}{\partial \dot{q}^a}\right)\frac{\partial^2 g^a}{\partial \dot{q}^\beta \partial \dot{q}^\alpha}\right) \neq 0$$

equation (7) determines uniquely the components f^α of the constrained SODE Γ.

This procedure is equivalent to considering first the free SODE $\tilde{\Gamma}_{\tilde{L}}$ defined by the Lagrangian $\tilde{\boldsymbol{L}}$ on $J^1(E)$, and then determining the corresponding constrained SODE Γ by projecting $\tilde{\Gamma}_{\tilde{L}}$ on the constraints manifold Σ [4].

Our purpose is to generalize this situation to the case of a SODE which is defined on Σ only. To this end, we introduce the following.

Definition 3.1. A SODE Γ, defined on Σ, is called Lagrangian if there exists a pair $(\boldsymbol{L}, \mu)$, where $\boldsymbol{L} \in \mathcal{C}^\infty(\Sigma)$ and $\mu = \mu_a \eta^a$, s.t.

$$i_\Gamma \mathrm{d}\Theta \in \mathrm{Span}\{\eta^a\} \tag{8}$$

where i_Γ is the interior product and Θ is given by

$$\Theta = \frac{\partial \boldsymbol{L}}{\partial \dot{q}^\alpha}\theta^\alpha + \mu + \boldsymbol{L}\,\mathrm{d}t. \tag{9}$$

Under the stated assumptions, the pair $(\boldsymbol{L}, \mu)$ will be called a *non-holonomic Lagrangian* for Γ. The 1-form (9) will be similarly called the *non-holonomic Poincaré Cartan 1-form* associated with $(\boldsymbol{L}, \mu)$.

From here on, we shall omit the word non-holonomic whenever there is no risk of ambiguity.

In local coordinates, condition (8) takes the form

$$\Gamma\left(\frac{\partial \boldsymbol{L}}{\partial \dot{q}^\alpha}\right) - \frac{\partial \boldsymbol{L}}{\partial q^\alpha} - B^a_\alpha \frac{\partial \boldsymbol{L}}{\partial q^a} = \mu_a Q^a_\alpha \tag{10}$$

which generalizes (7) when the Lagrangian $\boldsymbol{L}$ is not defined on the whole space $J^1(E)$.

Exactly as before, if the regularity hypothesis

$$\det\left(\frac{\partial^2 \boldsymbol{L}}{\partial \dot{q}^\beta \partial \dot{q}^\alpha} - \mu_a \frac{\partial^2 g^a}{\partial \dot{q}^\beta \partial \dot{q}^\alpha}\right) \neq 0 \tag{11}$$

is satisfied, and equation (10) determines uniquely the components f^α for the constrained SODE Γ.

If Γ is a non-holonomic Lagrangian SODE on Σ, we can extend the idea of Noether symmetries, in the following way.

Definition 3.2. Let Γ be a SODE associated to the non-holonomic Lagrangian $(\boldsymbol{L}, \mu)$, and let Θ be the corresponding non-holonomic Poincaré Cartan 1-form. A vector field $X \in \mathcal{D}^1(\Sigma)$ is a *Noether vector field* for Γ iff it satisfies $\mathcal{L}_X(\Theta) = \mathrm{d}f$ where $f \in \mathcal{C}^\infty(\Sigma)$.

We remark that a Noether vector field is not a dynamical symmetry, due to the fact that $\mathrm{d}\Theta$ is not a simplectic 2-form on Σ. The term 'Noether' is motivated by the fact that to a (particular class of) Noether vector fields one can associate a corresponding set of constants of motion by the following.

Theorem 3.1. If X is a Noether vector field for Γ satisfying $i_X(\eta) = 0 \forall \eta \in \mathrm{Span}\{\eta^a\}$, then $i_X(\Theta) - f$ is a constant of motion for Γ.

Proof. Let X be a Noether vector field, by definition 3.2 we have that $i_X(\mathrm{d}\Theta) = \mathrm{d}(f - i_X(\Theta))$. Contracting with Γ, and using the equation of motion $i_\Gamma(\mathrm{d}\Theta) \in \mathrm{Span}\{\eta^a\}$, we have

$$i_\Gamma(i_X(\mathrm{d}\Theta)) = -i_X(i_\Gamma(\mathrm{d}\Theta)) = 0 = \Gamma(f - i_X(\Theta)).$$

□

The supplementary condition $i_X(\eta) = 0$, required in order for a Noether vector field to generate a constant of motion, is by no means an artificial one. For example, in the study of Noether symmetries for non-conservative mechanical systems, one has to look for vector fields X having a vanishing pairing with the 1-form representing the non-conservative forces. In our case the role of non-conservative forces is played by reactive forces, which in principle are unknown, but can be described by 1-forms belonging to the $\mathrm{Span}\{\eta^a\}$.

Theorem 3.2. Let X be a vector field and Γ be a SODE on Σ associated to a non-holonomic Lagrangian $(\boldsymbol{L}, \mu)$. Then X is a Noether vector field for Γ iff $\alpha = i_X(\mathrm{d}\Theta)$ is a closed 1-form.

Proof. By using $\mathcal{L}_X(\mathrm{d}\Theta) = i_X(\mathrm{dd}\Theta) + \mathrm{d}(i_X(\mathrm{d}\Theta)) = \mathrm{d}\alpha$, the conclusion follows immediately. □

4. Helmholtz conditions

In the previous section we introduced the idea of Lagrangian constrained SODE. The purpose of this section is to characterize this special class of SODE over Σ. The analysis extends the approach proposed in [1] for the non-constrained case.

Theorem 4.1. Let Γ be a SODE on Σ. Then Γ is Lagrangian, with non-holonomic associated Lagrangian $(\boldsymbol{L}, \mu = \mu_a \eta^a)$, iff there exists a 1-form $\varphi = a_\alpha \theta^\alpha + \mu_a \eta^a + h\mathrm{d}t$ and a function $\boldsymbol{L} \in \mathcal{C}^\infty(\Sigma)$ such that

$$\mathcal{L}_\Gamma \varphi = \mathcal{L}_\Gamma(a_\alpha \theta^\alpha + \mu + h\mathrm{d}t) = \mathrm{d}\boldsymbol{L} + \nu \tag{12}$$

holds, where $\nu \in \mathrm{Span}\{\mathrm{d}t, \eta^a\}$.

Proof. $\Longleftarrow$ Using the local base (3), and considering the components along θ^α and ϕ^α of equation (12), we have

$$a_\alpha = \frac{\partial \boldsymbol{L}}{\partial \dot{q}^\alpha} \tag{13}$$

$$\Gamma(a_\alpha) - \frac{\partial \boldsymbol{L}}{\partial q^\alpha} - B^a_\alpha \frac{\partial \boldsymbol{L}}{\partial q^a} = \mu_a Q^a_\alpha. \tag{14}$$

It is easy to check that these two equations are equivalent to

$$i_\Gamma(\mathrm{d}\Theta) \in \mathrm{Span}\{\eta^a\}$$

where $\Theta = \frac{\partial \boldsymbol{L}}{\partial \dot{q}^\alpha}\theta^\alpha + \mu_a\eta^a + \boldsymbol{L}\mathrm{d}t$.

$\Longrightarrow$ It is a straightforward computation, after setting $\varphi = \Theta$. □

We shall now discuss a particular class of constrained SODE for which it is possible to extend some of the results valid in the non-constrained case. To this end, we prove the following.

Theorem 4.2. Let Γ be a constrained SODE on Σ. Then the following conditions are locally equivalent:

(1) there is a Lagrangian $(\boldsymbol{L}, \mu)$ for Γ such that $i_\Gamma(\mathrm{d}\Theta) = 0$,

(2) there is a closed 2-form ω such that $\mathcal{L}_\Gamma\omega = 0$ and $\omega(V, W) = 0 \forall V, W \in \mathrm{Span}\{\frac{\partial}{\partial \dot{q}^\alpha}\}$,

(3) there is a 1-form $\varphi = a_\alpha\theta^\alpha + \mu_a\eta^a + h\mathrm{d}t$ and a function $\boldsymbol{L} \in \mathcal{C}^\infty(\Sigma)$ such that $\mathcal{L}_\Gamma\varphi = \mathrm{d}\boldsymbol{L}$.

Proof. (1) $\Longrightarrow$ (2) With the choice $\omega = \mathrm{d}\Theta$, ω is indeed a closed 2-form satisfying $\mathcal{L}_\Gamma(\mathrm{d}\Theta) = \mathrm{d}(i_\Gamma(\mathrm{d}\Theta)) + i_\Gamma\mathrm{d}(\mathrm{d}\Theta) = 0$. Moreover

$$\mathrm{d}\Theta(V, W) = V(i_W\Theta) - W(i_V\Theta) - i_{[V,W]}\Theta = 0$$

because $i_W\Theta = 0, \forall W \in \mathrm{Span}\{\frac{\partial}{\partial \dot{q}^\alpha}\}$ and $[V, W] \in \mathrm{Span}\{\frac{\partial}{\partial \dot{q}^\alpha}\}, \forall W, V \in \mathrm{Span}\{\frac{\partial}{\partial \dot{q}^\alpha}\}$.

(2) $\Longrightarrow$ (3) The closedness of ω ensures that (locally) there exists a 1-form ψ such that $\omega = \mathrm{d}\psi$. Moreover, due to the condition $\omega(V, W) = 0, \forall V, W \in \mathrm{Span}\{\frac{\partial}{\partial \dot{q}^\alpha}\}$, the restriction of ψ to the vertical fibres is (locally) an exact 1-form. It is therefore possible to find a function $F \in \mathcal{C}^\infty(\Sigma)$ such that $\mathrm{d}F(V) = \psi(V), \forall V \in \mathrm{Span}\{\frac{\partial}{\partial \dot{q}^\alpha}\}$. By defining $\varphi = \psi - \mathrm{d}F$, we have that $\mathrm{d}\varphi = \mathrm{d}\psi = \omega$ and $\varphi(V) = 0, \forall V \in \mathrm{Span}\{\frac{\partial}{\partial \dot{q}^\alpha}\}$. Then φ can be written in the following form

$$\varphi = a_\alpha\theta^\alpha + \mu_a\eta^a + h\,\mathrm{d}t.$$

Moreover $0 = \mathcal{L}_\Gamma\omega = \mathcal{L}_\Gamma(\mathrm{d}\varphi) = \mathrm{d}\mathcal{L}_\Gamma\varphi$, and this guarantees the existence of a function $\boldsymbol{L} \in \mathcal{C}^\infty(\Sigma)$ such that (locally) $\mathcal{L}_\Gamma\varphi = \mathrm{d}\boldsymbol{L}$.

(3) $\Longrightarrow$ (1) Setting $\nu = 0$, theorem 4.1 guarantees that $(\boldsymbol{L}, \mu = \mu_a\eta^a)$ is a non-holonomic Lagrangian for Γ. Moreover, since $\varphi - \Theta = (h - \boldsymbol{L})\mathrm{d}t$, with $\Gamma(h - \boldsymbol{L}) = 0$, we have that $\mathcal{L}_\Gamma\varphi = \mathcal{L}_\Gamma\Theta$, and consequently $i_\Gamma(\mathrm{d}\Theta) = \mathcal{L}_\Gamma\Theta - \mathrm{d}(i_\Gamma(\Theta)) = \mathrm{d}\boldsymbol{L} - \mathrm{d}\boldsymbol{L} = 0$. □

Corollary 4.1. Let Γ be a SODE satisfying the conditions of theorem 4.2, and let X be a dynamical symmetry for Γ. Then the 1-form $\alpha = i_X(\mathrm{d}\Theta)$ is a dual symmetry such that $\mathcal{L}_\Gamma(\alpha) = 0$.

Corollary 4.2. Let Γ be a SODE satisfying the conditions of theorem 4.2, and let X be a dual-adjoint symmetry for Γ. Then the 1-form $\alpha = i_X(A\mathrm{d}\Theta)$ is an adjoint symmetry such that $\mathcal{A}_\Gamma(\alpha) = 0$.

Proof. A straightforward calculation shows that $i_X(\sigma) = A(i_{AX}(A\sigma))$, $\forall X \in \mathcal{D}^1(\Sigma)$, $\forall\sigma \in \mathcal{D}_1(\Sigma)$. If X is a dual-adjoint symmetry, then, by theorem 2.3, there is a dynamical symmetry Y such that $X = AY$. Then $i_X(A\mathrm{d}\Theta) = i_{AY}(A\mathrm{d}\Theta) = A(i_Y(\mathrm{d}\Theta))$. Since $i_Y(\mathrm{d}\Theta)$ is a dual symmetry satisfying $\mathcal{L}_\Gamma(i_Y(\mathrm{d}\Theta)) = 0$, then by theorem 2.3 $i_X(A\mathrm{d}\Theta)$ is an adjoint symmetry such that $\mathcal{A}_\Gamma(\alpha) = 0$. □

Corollary 4.3. Let Γ be a SODE satisfying the conditions of theorem 4.2, and let X be a Noether vector field for Γ, i.e. $\mathcal{L}_X(\Theta) = \mathrm{d}f$, where $f \in \mathcal{C}^\infty(\Sigma)$. Then $f - i_X(\Theta)$ is a constant of motion for Γ.

As a concluding remark, we shall now characterize two particular cases of constrained SODE with a non-holonomic Lagrangian, for which the dynamics is determined by the function $\boldsymbol{L}$ only, or more precisely by the pair $(\boldsymbol{L}, \mu = 0)$.

Theorem 4.3. Let Γ be a SODE on Σ. Assume $\tilde{f} \in \mathcal{C}^\infty(J^1(M))$ and let $f = \pi_2^*(\tilde{f})$ be the pull-back of $\tilde{f}$ on Σ. Putting $\sigma = \frac{\partial f}{\partial \dot{q}^\alpha}\theta^\alpha + f\mathrm{d}t$, we have that Γ is Lagrangian with non-holonomic Lagrangian $(\boldsymbol{L} = \Gamma(f), 0)$ iff $\mathcal{A}_\Gamma(\mathcal{L}_\Gamma(\sigma)) \in \mathrm{Span}\{\mathrm{d}t, \eta^a\}$.

Proof. A straightforward but tedious calculation shows that the 1-form $\mathcal{A}_\Gamma(\mathcal{L}_\Gamma(\sigma))$ has no components along ϕ^α, and that the vanishing of the components along θ^α gives

$$\Gamma\left(\frac{\partial \boldsymbol{L}}{\partial \dot{q}^\alpha}\right) - \frac{\partial \boldsymbol{L}}{\partial q^\alpha} - B_\alpha^a \frac{\partial \boldsymbol{L}}{\partial q^a} = 0 \tag{15}$$

where we used that $\frac{\partial \boldsymbol{L}}{\partial q^a} = \frac{\partial f^\beta}{\partial q^a}\frac{\partial \boldsymbol{L}}{\partial \dot{q}^\beta}$. Equation (15) has precisely the form of the equation of motion (10), when the 1-form μ vanishes. □

Theorem 4.4. A SODE Γ on Σ is Lagrangian, with associated non-holonomic Lagrangian $(\boldsymbol{L}, 0)$, iff there exists a function $\boldsymbol{L} \in \mathcal{C}^\infty(\Sigma)$ such that $\mathcal{A}_\Gamma(\mathrm{d}\boldsymbol{L}) \in \mathrm{Span}\{\mathrm{d}t, \eta^a, \theta^\alpha\}$

Proof. By definition of $\mathcal{A}_\Gamma$ we have

$$\mathcal{A}_\Gamma(\mathrm{d}\boldsymbol{L}) = A\mathcal{L}_\Gamma\left(\Gamma(\boldsymbol{L})\,\mathrm{d}t - H_\alpha(\boldsymbol{L})\theta^\alpha + \frac{\partial \boldsymbol{L}}{\partial q^a}\eta^a + \frac{\partial \boldsymbol{L}}{\partial \dot{q}^\alpha}\phi^\alpha\right).$$

The requirement that all components along the ϕ^α of the previous expression vanish yields to the condition

$$\Gamma\left(\frac{\partial \boldsymbol{L}}{\partial \dot{q}^\alpha}\right) - \frac{\partial \boldsymbol{L}}{\partial q^\alpha} - B_\alpha^a \frac{\partial \boldsymbol{L}}{\partial q^a} = 0.$$

□

The geometrical framework introduced in the previous sections may be conveniently applied in the study of mixed first- and second-order systems of differential equations (see, e.g. [12]).

To pursue this idea, suppose that a SODE Γ of the form (1) is given. The latter corresponds to the mixed system of differential equations

$$\ddot{q}^\alpha = f^\alpha(t, q^A, \dot{q}^\alpha) \tag{16a}$$

$$\dot{q}^a = g^a(t, q^A, \dot{q}^\alpha). \tag{16b}$$

In this context an interesting problem is to establish under what conditions the second-order equations (16*a*) and the first order equations (16*b*) can be decoupled. Making use of the differential operator $\mathcal{A}_\Gamma$, a useful result is provided by the following.

Theorem 4.5. In the system (16) associated to a SODE Γ of the form (1), the second-order equations and the first-order equations can be decoupled, and the second order equations are deducible from a Lagrangian $\boldsymbol{L}$ (in the usual sense), iff there exists a function $\boldsymbol{L} \in \mathcal{C}^\infty(J^1(M))$ satisfying the regularity condition $\det\left(\frac{\partial^2 \boldsymbol{L}}{\partial \dot{q}^\alpha \partial \dot{q}^\beta}\right) \neq 0$, such that

$$\mathcal{A}_\Gamma(\mathrm{d}\boldsymbol{L}) \in \mathrm{Span}\{\mathrm{d}t, \theta^\alpha, \eta^a\}.$$

Proof. From equation (15), by using the condition $\frac{\partial \boldsymbol{L}}{\partial q^a} = 0$, we have that $\mathcal{A}_\Gamma(\mathrm{d}\boldsymbol{L}) \in \mathrm{Span}\{\mathrm{d}t, \theta^\alpha, \eta^a\}$ iff

$$\Gamma\left(\frac{\partial \boldsymbol{L}}{\partial \dot{q}^\alpha}\right) - \frac{\partial \boldsymbol{L}}{\partial q^\alpha} = 0.$$

Moreover, by using the regularity hypothesis on $\boldsymbol{L}$ we get the expression $f^\alpha = f^\alpha(t, q^\beta, \dot{q}^\beta)$, and $g^a = g^a(t, q^A, \dot{q}^\alpha)$. □

Remark 4.1. The inverse problem for a system of mixed equations (of first and second order) was approached in a completely different way in [3]: the idea there was to look for a singular Lagrangian giving the first-order equations as Dirac constraints, and the second-order equations as the equations related to the regular part of the Lagrangian.

5. Example

We conclude this paper with an illustrative example. More precisely, we show the existence of a non-conservative holonomic mechanical system which, by imposing a suitable kinetic constraint, is Lagrangian in the sense introduced in section 3.

Example 5.1. A rigid frame ABO, composed of two homogeneous bars $\overline{AB}$ and $\overline{OB}$, with respective lengths L and $\frac{L}{2}$ and masses M and $\frac{M}{2}$, soldered in B at an angle $A\hat{B}O = \frac{\pi}{3}$, is constrained to a vertical axis $\boldsymbol{k}_3$ by means of two hinges placed in A and O. A material point P, with mass $m = \frac{M}{2}$, can move along the side $\overline{AB}$ of the frame. All constraints are ideal. The configurations of the system are described by two Lagrangian coordinates ψ and s expressing respectively the rotation of the frame around the vertical axis and the distance $\overline{PA}$. In addition to the weights $M\boldsymbol{g}$ and $m\boldsymbol{g}$, the system is subject to two further forces $\boldsymbol{F}$ and $\boldsymbol{G}$, both acting on P, and expressed respectively by the equations

$$\boldsymbol{G} = m\boldsymbol{\omega} \wedge \boldsymbol{v}_P$$
$$\boldsymbol{F} = -\frac{m}{2}(\dot{s} - l\dot{\psi})\boldsymbol{\omega} \wedge (\cos\psi\, \boldsymbol{k}_1 + \sin\psi\, \boldsymbol{k}_2)$$

where $\boldsymbol{\omega} = \dot{\psi}\boldsymbol{k}_3$ is the angular velocity of the frame, $\boldsymbol{v}_P$ is the velocity of P and l is a suitable constant coefficient. The system described above is holonomic, non-conservative with holonomic kinetic energy

$$\tilde{\boldsymbol{T}} = \frac{m}{2}\dot{s}^2 + \frac{m}{8}(s^2 + L^2)\dot{\psi}^2.$$

Taking into account the force of gravity acting on the particle moving along AB, we can consider the Lagrangian $\tilde{\boldsymbol{L}} = \tilde{\boldsymbol{T}} - mg\frac{\sqrt{3}}{2}(L - s)$ and the Lagrangian components of the forces $\boldsymbol{G}$ and $\boldsymbol{F}$ given by the expressions

$$\mathcal{Q}_\psi = +\frac{m}{4}ls\dot{\psi}^2$$
$$\mathcal{Q}_s = -\frac{m}{4}s\dot{\psi}^2.$$

Now we assume that the system is subject to the kinetic constraint

$$\dot{s} = \nu\psi \qquad \nu = \text{constant}.$$

The effect of this constraint is to make the velocity of P along $\overline{AB}$ proportional to the angle of rotation ψ. The constrained SODE Γ on Σ representing the dynamic of the constrained system may be obtained in various ways (see, for example, [4, 6, 9, 11]). The final result is expressed by the equation

$$\Gamma = \frac{\partial}{\partial t} + \dot{\psi}\frac{\partial}{\partial \psi} + \nu\psi\frac{\partial}{\partial s} - \frac{2s\nu\psi\dot{\psi} - ls\dot{\psi}^2}{s^2 + L^2}\frac{\partial}{\partial \dot{\psi}}.$$

Recalling (2) and (3), the local bases of $T\Sigma$ and $T^*\Sigma$ induced by Γ are

$$\Gamma \qquad H = \frac{\partial}{\partial \psi} - \frac{s\nu\psi - ls\dot{\psi}}{s^2 + L^2}\frac{\partial}{\partial \dot{\psi}}, \frac{\partial}{\partial s}, \frac{\partial}{\partial \dot{\psi}}$$

and

$$\mathrm{d}t, \theta = \mathrm{d}\psi - \dot{\psi}\,\mathrm{d}t, \eta = \mathrm{d}s - \nu\psi\,\mathrm{d}t,$$
$$\phi = \mathrm{d}\dot{\psi} + \frac{2s\nu\psi\dot{\psi} - ls\dot{\psi}^2}{s^2 + L^2}\mathrm{d}t + \frac{s\nu\psi - ls\dot{\psi}}{s^2 + L^2}\theta.$$

In terms of these bases it is easy to see that the pair $(\boldsymbol{L}, \mu)$, where

$$\boldsymbol{L} = i^*(\tilde{\boldsymbol{L}}) = i^*\left(\tilde{\boldsymbol{T}} - mg\frac{\sqrt{3}}{2}(L - s)\right)$$

and

$$\mu = i^*\left(\frac{\partial \tilde{\boldsymbol{T}}}{\partial \dot{s}} + \frac{l}{\nu}\mathcal{Q}_s\right)\eta$$

is a non-holonomic Lagrangian for the SODE Γ. In fact, given the non-holonomic Poincaré Cartan 1-form

$$\Theta = \frac{\partial \boldsymbol{L}}{\partial \dot{\psi}}\theta + \mu + \boldsymbol{L}\,\mathrm{d}t$$

the equation of motion for the constrained system can be written as

$$i_\Gamma \mathrm{d}\Theta \in \mathrm{Span}\{\eta\}$$

or in the equivalent form

$$\Gamma\left(\frac{\partial \boldsymbol{L}}{\partial \dot{\psi}}\right) - \frac{\partial \boldsymbol{L}}{\partial \psi} = -\nu i^*\left(\frac{\partial \tilde{\boldsymbol{T}}}{\partial \dot{s}} + \frac{l}{\nu}\mathcal{Q}_s\right).$$

We remark that, since the potential of the force of gravity does not affect the equation of motion on Σ, we can delete it in the expression of $\boldsymbol{L}$. Indeed the function $\boldsymbol{L}$ is defined modulo a function depending on the s-variable only.

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